6.10.5 Systems of Linear ODEs

See also:

In this 19 minute video MIT professor Gilbert Strang explains how eigenvectors and eigenvalues give us the solution to a system of first order, linear ordinary differential equations.

Differential equations that come up in engineering design and analysis are usually systems of equations, rather than a single equation. Fortunately, they are often first order linear equations. As discussed in the *Symbolic Differential Equations* section, the Symbolic Math Toolbox can solve many differential equations expressed by a single equation. Higher order and non-polynomial systems of ODEs need numerical methods, such as discussed in the *Numeric Differential Equations* section. However, systems of first order linear ODEs may be solved analytically with eigenvalues and eigenvectors.

Equations with exponents of the special number e have the special property that it is the only function whose derivative is a scalar multiple of itself. Specifically,

$$\boxed{\frac{d\,e^{a\,t}}{dt} = a\,e^{a\,t}.}$$

Thus, it follows that ODEs of the form

$$\frac{dy(t)}{dt} = a\,y(t)$$

have the solution

$$y(t) = c\,e^{a\,t}.$$

Note: Do you see why the derivative of $e^{a\,t}$ is a scalar multiple of itself? If we don't use the known derivative of $e^{a\,t}$, we can either take the derivative of its Maclaurin (Taylor) series, or use it numeric definition in terms of a limit. I will use the later.

$$e^{a\,t} = \lim_{n\to\infty}\left(1 + \frac{a\,t}{n}\right)^n$$

You need to use the chain rule to take the derivative. If $f(t) = \left(1 + \frac{a\,t}{n}\right)^n$, then $f'(t) = a\,\left(1 + \frac{a\,t}{n}\right)^{n-1}$. We see the desired equality then in the limit.

$$\begin{aligned} e^{a\,t} &= \lim_{n\to\infty} f(t) \\ \frac{d}{dt}\left(e^{a\,t}\right) &= \lim_{n\to\infty} f'(t) = a\,\lim_{n\to\infty} f(t) = a\,e^{a\,t} \end{aligned}$$

The same principle applies to systems of ODEs, except that we use vectors and matrices to describe the equations.

$$\begin{cases} y_1' = a_{11}\,y_1 + a_{12}\,y_2 + \cdots + a_{1n}\,y_n \\ y_2' = a_{21}\,y_1 + a_{22}\,y_2 + \cdots + a_{2n}\,y_n \\ \quad\vdots \\ y_n' = a_{n1}\,y_1 + a_{n2}\,y_2 + \cdots + a_{nn}\,y_n \end{cases}$$

$y = e^{at}c$

$\vec{y} = e^{At}\vec{k}$

$c = X^{-1}K$

In matrix notation, this is

$$\boldsymbol{y}' = \boldsymbol{A}\,\boldsymbol{y}.$$

The solution has the form

$$\boldsymbol{y(t)} = c_1 e^{\lambda_1 t}\boldsymbol{x_1} + c_2 e^{\lambda_2 t}\boldsymbol{x_2} + \cdots + c_n e^{\lambda_n t}\boldsymbol{x_n}$$

The set of scalar values $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ are the eigenvalues of matrix $\boldsymbol{A}$. The vectors $\{\boldsymbol{x_1}, \boldsymbol{x_2}, \cdots, \boldsymbol{x_n}\}$ are the eigenvectors of $\boldsymbol{A}$.

After we learn about *Diagonalization and Powers of A*, we will have seen enough linear algebra to see where this solution comes from. The solution is derived in the appendix under section *A Matrix Exponent and Systems of ODEs*.

ODE Example

Consider the set of ODEs and initial conditions,

$$\begin{cases} y_1(t)' = -2\,y_1(t) + \quad y_2(t) \\ y_2(t)' = \qquad y_1(t) - 2\,y_2(t) \end{cases}, \qquad \begin{cases} y_1(0) = 6 \\ y_2(0) = 2 \end{cases}.$$

In matrix notation,

$$\boldsymbol{y}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\boldsymbol{y}, \qquad y(0) = \begin{bmatrix} 6 \\ 2 \end{bmatrix}.$$

We first use MATLAB to find the eigenvalues and eigenvectors. MATLAB always returns normalized eigenvectors, which can be multiplied by a constant to get simpler numbers.

```
>> A = [-2 1; 1 -2];
>> [X,lambda] = eig(A)
X =
    0.7071    0.7071
   -0.7071    0.7071
lambda =
    -3     0
     0    -1
>> X = X*2/sqrt(2)
X =
    1.0000    1.0000
   -1.0000    1.0000
```

The columns of the `X` matrix are the eigenvectors. The eigenvalues are on the diagonal of `lambda` . Our solution has the form

$$\boldsymbol{y(t)} = c_1\, e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\, e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$c = X^{-1} \vec{y}(0)$

At the initial condition, the exponent terms become 1.

$$\boldsymbol{y}(\boldsymbol{0}) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \boldsymbol{X\, c}$$

```
>> y0 = [6;2];
>> c = X\Y0
c =
    2.0000
    4.0000
```

$$\begin{cases} y_1(t) = 2\, e^{-3t} + 4\, e^{-t} \\ y_2(t) = -2\, e^{-3t} + 4\, e^{-t} \end{cases}$$

Note: Some ODE systems have complex eigenvalues. When this occurs, the solution will have sine and cosine oscillating terms because of Euler's formula, $e^{j\,x} = \cos(x) + j\, \sin(x)$.

Stability λ - real , complex - oscillation

$\lambda < 0$