6.10 Application of Eigenvalues and Eigenvectors

Eigenvalue / Eigenvector problems are one of the more important linear algebra topics. Eigenvalue and eigenvectors are used to solve systems of differential equations, but more generally they are used for data analysis, where the matrix $\mathbf{A}$ represents data rather than coefficients of a system of equations. They introduce a simple, yet very powerful relationship between a matrix and a set of special vectors and scalar values. This simple relationship provides elegant solutions to some otherwise difficult problems.

6.10.1 Introduction to Eigenvalues and Eigenvectors

Video Resource

In this lecture, Professor Gilbert Strang of MIT gives a good introduction to eigenvalues and eigenvectors.

The Best Kept Secret of Engineering

Whether you know it or not, you use eigenvalues and eigenvalues everyday. Furthermore, knowing about them can greatly expand your ability to solve interesting and challenging problems. Engineers encounter eigenvalues and eigenvectors when studying mechanics, vibrations, or when working with big data. Google search uses eigenvectors to rank pages, and Netflix uses eigenvectors to predict your preference for a movie you have not yet watched. Since eigenvectors and eigenvalues are not well known by most people that are not engineers, scientist, or mathematicians, many may not realize how useful they are. In actuality, they are useful for many problems.

We use eigenvalues and eigenvectors to better understand and simplify expressions involving a matrix. We can not, of course, say that a matrix is equivalent to a single scalar value, but when multiplying a matrix by one of its eigenvectors, then the matrix may be replaced by a scalar (its eigenvalue). **What an incredibly fantastic simplification!** Many problems allow us to represent a matrix in terms of its eigenvalues and eigenvectors. Making this replacement can reduce the computational complexity of a problem and reveal valuable insights about the matrix and the problems we are attempting to solve.

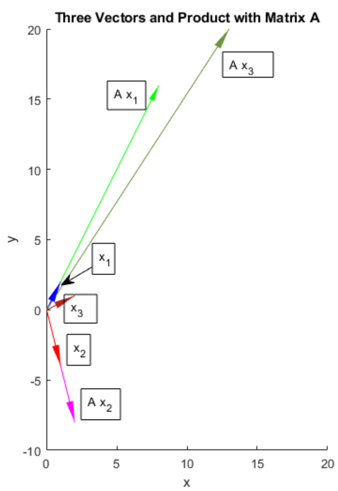
Admittedly, you may need to think about this for a while and see some application examples before appreciating the full value of eigenvectors and eigenvalues. The important thing to remember is that eigenvectors and eigenvalues reveal and take advantage of important properties of matrices. What's more is that using a computer to find the eigenvectors and eigenvalues makes them easy to use and apply to various problems.

Matrix multiplication is like yoga to a vector – mostly stretching and rotation. That is, the most common product of a matrix and a vector ($\boldsymbol{A}\,\boldsymbol{x}$) results in a vector that is a stretch and rotation of the original vector $\boldsymbol{x}$. However, that is not always the case. There are special matrices that rotate a vector, but do not change its length. (See *Rotation of a Point* for more on rotation matrices.) There are also vectors corresponding to each square matrix that will be stretched by the multiplication, but not rotated. This later case is what we consider here.

Consider the following matrix and three vectors.

$$A = \begin{bmatrix} 6 & 1 \\ 8 & 4 \end{bmatrix},\; x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\; x_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix},\; x_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The plot below shows the three vectors and the product of each vector with matrix $\boldsymbol{A}$. The product $\boldsymbol{A}\,\boldsymbol{x_3}$ shows the typical result of matrix to vector multiplication. The multiplication both stretches and rotates the vector. But in the cases of $\boldsymbol{A}\,\boldsymbol{x_1}$ and $\boldsymbol{A}\,\boldsymbol{x_2}$, the vectors are stretched, but not rotated. This is because vectors $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ are eigenvectors to matrix $\boldsymbol{A}$.



When a vector $\boldsymbol{x_i}$ is one of the eigenvectors of a matrix $\boldsymbol{A}$, the following special relationship holds:

$$\boxed{\boldsymbol{A}\,\boldsymbol{x_i} = \lambda_i\,\boldsymbol{x_i}}$$

For the n-by-n square matrix, there are n *eigenvectors*, $\boldsymbol{x_i}$, and n corresponding scalar *eigenvalues*, λ_i.

That is to say that when one of the eigenvectors is multiplied by the $\boldsymbol{A}$ matrix, only stretching occurs – no rotation. The eigenvalues account for the stretching (scaling) factor.

The power of this relationship is that when, and only when, we multiply a matrix by one of its eigenvectors, then in a simplified equation we can **replace a matrix with just a simple scalar value**. So to take advantage of the simplification, we like to multiply matrices by their eigenvectors wherever possible.

What's in the name?

In the German language, the word *eigen* means *own*, as in "my own opinion" or "my own family". It is also used for the English word *typical*, as in "that is typical of him". Perhaps the old German word *eigenschaft* meaning feature, property, or characteristic is more clear as to the intent. For a matrix, the eigenvectors and eigenvalues represent the vector directions and magnitudes distinctly embodied by the matrix.

From Wikipedia: In the 18th century Euler studied the rotational motion of a rigid body and discovered the importance of the principal axes. Lagrange realized that the principal axes are the eigenvectors of the inertia matrix. In the early 19th century, Cauchy saw how their work could be used to classify the quadric surfaces, and generalized it to arbitrary dimensions. Cauchy also coined the term *racine caracteristique* (characteristic root) for what is now called eigenvalue.

German Mathematician David Hilbert (1862 - 1943) is credited with naming them eigenvalues and eigenvectors.