CHAPTER

SIX

INTRODUCTION TO LINEAR ALGEBRA

As was discussed in our *Introduction to the Course*, linear algebra is perhaps the branch of mathematics that is most useful to engineers. However, it is often overshadowed by King Calculus. Early evidence of linear algebra occurred several thousand years ago. Swiss mathematician Leonhard Euler worked on fairly advance linear algebra concepts around 300 years ago. Yet even in the middle of the twentieth century, it was not in wide use by most engineers and scientists. The advent of the computer has been a boon to the popularity of linear algebra. Calculations on vectors and matrices often require a large number of simple multiplications and additions, which are tedious for humans but can be preformed quickly and accurately by computers. With wider use in recent years, we have discovered new applications for linear algebra in areas related to control systems, data analysis and artificial intelligence. It is highly appropriate to learn about linear algebra in a course focused on computational engineering rather than one focused on analytic mathematics. We will learn the basics of the math concepts, but we will quickly turn to the computer to do the heavy lifting of computing results.

Our coverage here of linear algebra in no way covers the depth of material found in a math course on linear algebra, such as Dr. Strang's free online course *[STRANG99]*. Our focus is distinctly applied to the computation of engineering problems.

Dr. Strang has provided a introduction to linear algebra: `LinAlg_nutshell.pdf` . If you'd like more information on vectors, matrices, matrix multiplication, and transforming vectors, look at the following Khan Academy videos:

- Vector introduction for linear algebra
- Introduction to matrices
- Introduction to matrix multiplication
- Introduction to identity matrix
- Transforming vectors using matrices

There are at least five very important applications of linear algebra to engineering problems:

1. Problems related to spacial vectors and geometry,
2. Solutions to systems of linear equations,

3. Vector projections with application to least squares regression and other optimization problems,
4. Solutions to systems of differential equations using eigenvalues and eigenvectors,
5. Other applications of eigenvalues and eigenvectors.

Note: On pages of the study guide where scalar variables are mixed with variables representing vectors and matrices, I will attempt to distinguish them by displaying vectors and matrices in a bold font while keeping scalars in a normal font.

Vector variables will be represented with lower-case letters and matrix variables will be upper-case letters.

6.1 Working with Geometric Vectors

We have already defined vectors and discussed how to create them and use them for may applications in the *Vectors and Matrices in MATLAB* section. Here we define some linear algebra operations on geometric vectors.

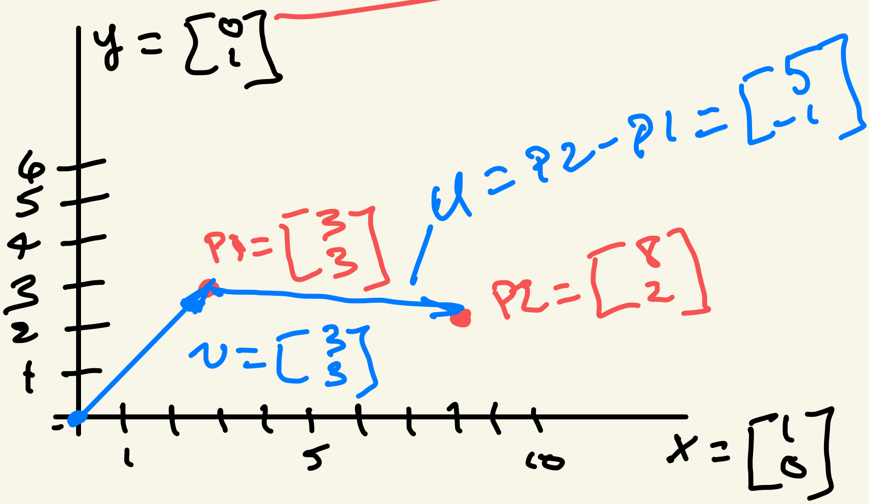
Vectors have scalar coefficients defining their displacement in each dimension of the coordinate axis system. Thus we can think of a vector as having a specific length and direction. What we don't always know about a vector is where it begins. Lacking other information about a vector we can consider that the vector begins at the origin. **Points** are sometimes confused for vectors because they are also defined by a set of coefficients for each dimension. The values of points are always relative to the origin. Vectors may be defined as spanning between two points; and one of the points may be the origin ($\boldsymbol{v} = p_2 - p_1$).

Note that in geometry and linear algebra, the standard usage of the term *vector* is that of a **column vector**. Because column vectors take more space when displayed, they are often displayed in one of two alternate ways – either as the transpose of a row vector ($[a\,b\,c]^T$) or with parentheses instead of square brackets ((a, b, c)). Although the later representation looks more like a row vector, it still should be regarded as a column vector for purposes of linear algebra.

6.1.1 Dimension and Space

We normally think of the term **dimension** as meaning how many values does a vector include, which is accurate for Euclidean geometry. Although in *Vector and Matrix Spaces*, we will explain a more precise definition as dimension being the number of basis vectors in the vector space. A **vector space** consists of a set of vectors and a set of scalars that are *closed* under vector addition and scalar multiplication. By saying that they are *closed* just means that we can add any vectors in the set together and multiply by any scalars in the set and the resulting vectors are still in the vector space.

Vectors



$$v = [x\ y]\begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3x + 3y$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

axis – basis vectors

$\mathbb{R}^2$ $\quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\quad\quad x \quad\quad y$

Span

$$w = 3\underline{\underline{x}} + 4\underline{\underline{y}}$$

$$A = \begin{bmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{bmatrix}$$

Vector Space – $\mathbb{R}^2$

$\mathbb{R}^3$

$\mathbb{R}^n$

$\mathbb{C}^n$

$R^2 \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0.4082 \\ 0.4082 \\ 0.8165 \end{bmatrix} \quad \begin{bmatrix} -0.4364 \\ 0.8729 \\ -0.2182 \end{bmatrix}$

$\|u\| = 1 \qquad \|v\| = 1$

$\perp$

Why column vectors

$$[A] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$Ax = b$$

For example, vectors in our physical 3-dimensional world are said to be in a vector space called $\mathbb{R}^3$. The set of vectors in $\mathbb{R}^3$ consist of three real numbers defining their magnitude in the $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ directions. The set of scalars in $\mathbb{R}^3$ is the set of real numbers. Similarly, the set of real scalars and the set of vectors on a 2-D plane, like a piece of paper, are said to be in the vector space called $\mathbb{R}^2$.

Other vector spaces may also be used for applications not relating to geometry and may have higher dimension than 3. Generally, we call this $\mathbb{R}^n$. For some applications, the coefficients of the vectors and scalars may also be complex numbers, which is a vector space denoted as $\mathbb{C}^n$.

6.1.2 Linear

Vectors are linear, which means that they can be added together ($\boldsymbol{u} + \boldsymbol{v}$) and stretched by being multiplied by a scalar constant ($k\,\boldsymbol{v}$). Vectors may also be defined as a linear combinations of other vectors ($\boldsymbol{x} = 3\,\boldsymbol{u} + 5\,\boldsymbol{v}$).

The set of all linear combinations of a collection of vectors is called the **span** of the vectors. So, the span of two vectors in $\mathbb{R}^3$ will be a plane. In the case where the vectors point in the same direction, the span will be a line.

6.1.3 Transpose

The transpose of either a vector or matrix is the reversal of the rows for the columns.

$$\text{Let } \boldsymbol{a} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ then, } \boldsymbol{a}^T = \begin{bmatrix} a & b & c \end{bmatrix}$$

$$\text{Let } \boldsymbol{a} = \begin{bmatrix} a & b & c \end{bmatrix} \text{ then, } \boldsymbol{a}^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\text{Let } \boldsymbol{C} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \text{ then, } \boldsymbol{C}^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

In MATLAB, the transpose operator is the apostrophe (`'`):

```
>> A_trans = A';
```

Note: To be completely accurate, we should point out that `.'` is the operator for a simple transpose, and `'` performs a complex conjugate transpose. For matrices with only real numbers, the result of the two operators is the same. For matrices with complex numbers, the complex conjugate is usually desired. For simplicity (or maybe laziness), I and everyone else tend to always use `'` for vectors of real numbers.

6.1.4 Dot Product

The sum of products between two vectors is called a dot product, or *inner product*. The operation yields a scalar.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} d \\ e \\ f \end{bmatrix} = a\,d + b\,e + c\,f$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32$$

The dot product operation is also defined in terms of a *sum of products* between a row vector by a column vector. That is to say, it is the same calculation used for each value of an inner product *Matrix Multiplication*, which we will cover shortly. The row vector is found by taking the transpose of the first vector.

For example,

$$\begin{aligned} \boldsymbol{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \\ \boldsymbol{a} \cdot \boldsymbol{b} &= \boldsymbol{a}^T\,\boldsymbol{b} = \begin{bmatrix} 1 & 2 \end{bmatrix} \times \begin{bmatrix} 3 \\ 4 \end{bmatrix} \\ &= 1 \cdot 3 + 2 \cdot 4 \\ &= 11 \end{aligned}$$

MATLAB has a `dot` function that takes two vector arguments and returns the scalar dot product. However, it is often just as easy to implement a dot product using a transpose and multiplication.

```
>> a = [3 5]'
a =
    3
    5
>> b = [2 4]'
b =
    2
    4
>> c = dot(a,b)
c =
    26
>> c = a'*b
```

```
c =
    26
```

Note: If the vectors are both row vectors (non-standard), then the dot product becomes $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a}\,\boldsymbol{b}^T$.

6.1.5 Dot Product Properties

Commutative

The dot product $\boldsymbol{u} \cdot \boldsymbol{v}$ equals $\boldsymbol{v} \cdot \boldsymbol{u}$. The order does not matter.

Length of Vectors

The *length* $\|\boldsymbol{v}\|$ of the vector $\boldsymbol{v}$ is the square root of the dot product of $\boldsymbol{v}$ with itself.

$$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = (v_1^2 + v_2^2 + ... + v_n^2)^{1/2}$$

MATLAB has a function called `norm` that takes a vector as an argument and returns the length.

A *normalized* or *unit* vector is a vector of length one. A vector may be normalized by dividing it by its length.

$$\tilde{\boldsymbol{v}} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$$

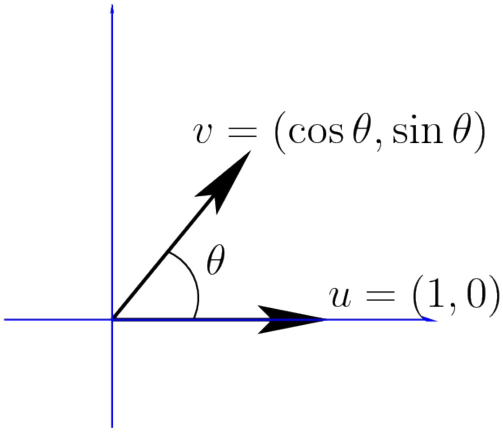
Angle Between Vectors

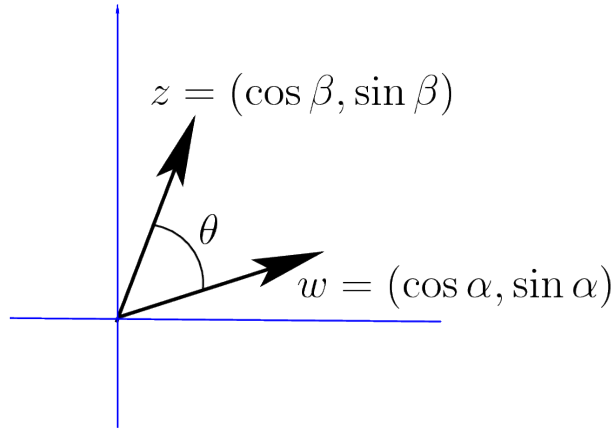
Consider two unit vectors (length 1) in $\mathbb{R}^2$, $\boldsymbol{u} = (1, 0)$ and $\boldsymbol{v} = (\cos\theta, \sin\theta)$. Then consider the same vectors rotated by an angle α, such that $\theta = \beta - \alpha$. Refer to a table of trigonometry identities for the final conclusion below.

$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{v} &= 1 \cdot \cos\theta + 0 \cdot \sin\theta = \cos\theta \\ \\ \boldsymbol{w} \cdot \boldsymbol{z} &= \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta \\ &= \cos(\beta - \alpha) \\ &= \cos\theta \end{aligned}$$

The previous result can also be found from the *law of cosines*, which tells us that

$$\|\boldsymbol{z} - \boldsymbol{w}\|^2 = \|\boldsymbol{z}\|^2 + \|\boldsymbol{w}\|^2 - 2\|\boldsymbol{z}\|\,\|\boldsymbol{w}\| \cos\theta$$





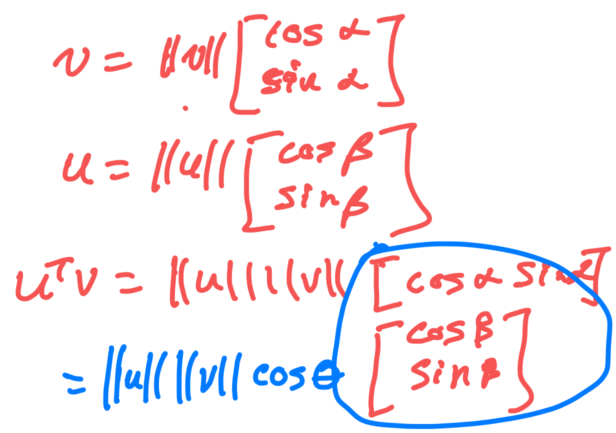
Since both $\boldsymbol{z}$ and $\boldsymbol{w}$ are unit vectors, several terms become 1.

$$\begin{aligned}(\boldsymbol{z}-\boldsymbol{w})\cdot(\boldsymbol{z}-\boldsymbol{w}) &= 2 - 2\,\cos\theta \\ \boldsymbol{z}\cdot\boldsymbol{z} - 2\,\boldsymbol{w}\cdot\boldsymbol{z} + \boldsymbol{w}\cdot\boldsymbol{w} &= 2 - 2\,\cos\theta \\ -2\,\boldsymbol{w}\cdot\boldsymbol{z} &= -2\,\cos\theta \\ \boldsymbol{w}\cdot\boldsymbol{z} &= \cos\theta\end{aligned}$$

When the vectors are not unit vectors, the vector lengths factor out as constants. A unit vector is obtained by dividing the vector by its length.

$$\left(\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\right)\cdot\left(\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|}\right) = \cos\theta$$

$$\boxed{\boldsymbol{v}\cdot\boldsymbol{w} = \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|\cos\theta}$$



All angles have $|\cos\theta| \leq 1$. So all vectors have:

$$|\boldsymbol{v}\cdot\boldsymbol{w}| \leq \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$$

Orthogonal Vector Test

From the previous property, one can easily see that when two vectors are perpendicular ($\theta = \pi/2$ or $90°$), their dot product is zero. This property extends to $\mathbb{R}^3$ and beyond, where we say that the vectors in $\mathbb{R}^n$ are orthogonal when their dot product is zero.

This is a **very important result**. The geometric properties of orthogonal vectors provide useful strategies for finding optimal solutions for some problems. An example of this is demonstrated in the *Over-determined Systems and Vector Projections* section.

`Perpendicular Vectors`

By *Pythagoras Law*, if $\boldsymbol{v}$ and $\boldsymbol{w}$ are perpendicular, then:

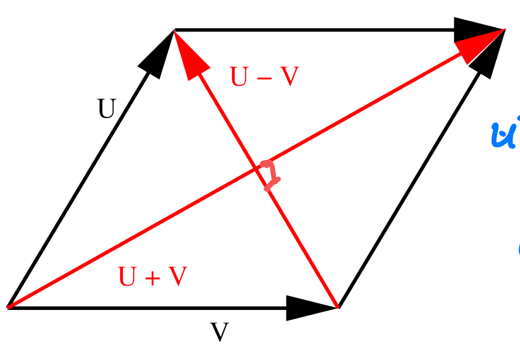
$$\|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 = \|\boldsymbol{v}-\boldsymbol{w}\|^2$$

```
>> v = [1 0]'
v =
     1
     0
>> w = [0 1]'
w =
     0
     1
>> z = v - w
z =
     1
    -1
>> v'*v + w'*w
ans =
     2
>> z'*z
ans =
     2
```

6.1.6 Application of Dot Products

Perpendicular Rhombus Vectors

A *rhombus* is any parallelogram whose sides are the same length. Use the properties of dot products to show that the diagonals of a rhombus are perpendicular.



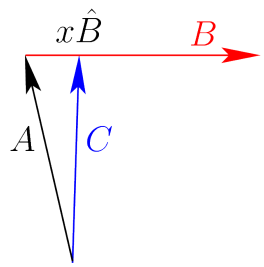
$$(u-v)^T(u+v) \stackrel{?}{=} 0$$
$$(u^T-v^T)(u+v) = 0$$
$$u^Tu + \underbrace{(u^Tv - v^Tu)}_{0} - v^Tv = 0$$
$$u^Tu - v^Tv = 0$$
$$u^Tu = v^Tv$$
$$\|u\|^2 \quad \|v\|^2$$

Fig. 6.1: Show that $(\boldsymbol{u}+\boldsymbol{v})\perp(\boldsymbol{u}-\boldsymbol{v})$

Hint: The Wikipedia page for transpose lists many transpose properties including the transpose with respect to addition, $(\boldsymbol{a}+\boldsymbol{b})^T = \boldsymbol{a}^T + \boldsymbol{b}^T$. Also, the fact that the sides of a rhombus are the same length is needed.

Find a Perpendicular Vector

Given two concatenated vectors in $\mathbb{R}^2$, $\boldsymbol{A}$, and $\boldsymbol{B}$ as shown below, derive an equation from vectors $\boldsymbol{A}$, and $\boldsymbol{B}$ to compute vector $\boldsymbol{C}$ such that $\boldsymbol{C}$ is perpendicular to $\boldsymbol{B}$ and when $\boldsymbol{C}$ begins at the origin of $\boldsymbol{A}$ it terminates at a point on $\boldsymbol{B}$ completing a right triangle.



You will want to identify the vector from the terminal point of $\boldsymbol{A}$ to the terminal point of $\boldsymbol{C}$ as $x\,\hat{\boldsymbol{B}}$ where x is a scalar and $\hat{\boldsymbol{B}}$ is a unit vector in the direction of $\boldsymbol{B}$.

You can find the answer either by starting with the orthogonality requirement between vectors $\boldsymbol{C}$ and $\hat{\boldsymbol{B}}$ or by using the trigonometry requirements for forming a right triangle.

The correct equation is either:

$$\boldsymbol{C} = \boldsymbol{A} - \boldsymbol{A^T}\,\hat{\boldsymbol{B}}\,\hat{\boldsymbol{B}}$$

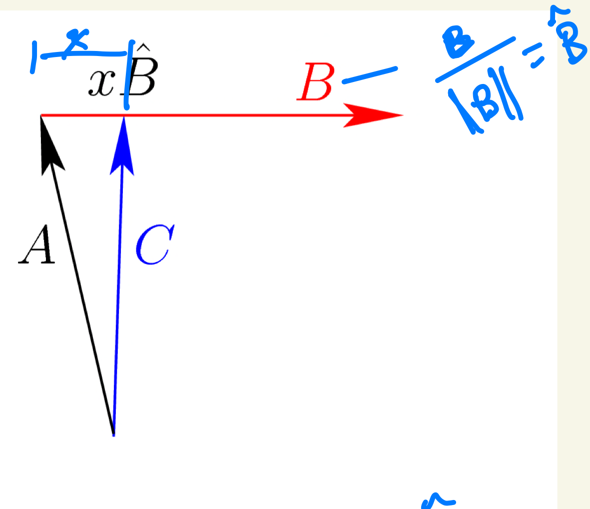
or

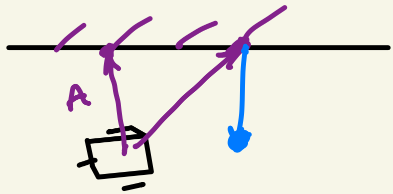
$$\boldsymbol{C} = \boldsymbol{A} - \hat{\boldsymbol{B}}\,\hat{\boldsymbol{B}}^{\boldsymbol{T}}\,\boldsymbol{A}$$

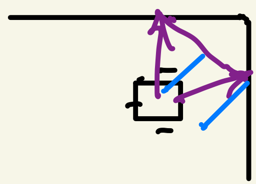
Can you see why these two equations are actually the same?

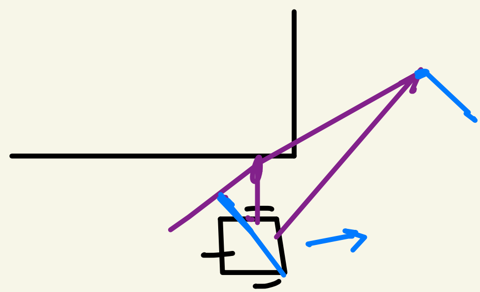
Verify your answer with example vectors in MATLAB. You might try the familiar right triangle with side lengths of 3, 4, and 5.

Mobile Robot Wall Following









$$C = A + \underline{x}\hat{B}$$

$$C^T \hat{B} = 0$$

$$(A + x\hat{B})^T \hat{B} = 0$$

$$(A^T + x\hat{B}^T)\hat{B} = 0$$

$$A^T B + x \underbrace{\hat{B}^T \hat{B}}_{1} = 0$$

$$x = -A^T \hat{B}$$

$$C = A - A^T\hat{B}\,\hat{B} = A - \hat{B}^T A \hat{B}$$

6.1.7 Outer Product

Whereas the inner product (dot product) of two vectors is a scalar ($\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^T\, \boldsymbol{v}$), the *outer product* of two vectors is a matrix.

$$\boldsymbol{u} \otimes \boldsymbol{v} = \boldsymbol{u}\, \boldsymbol{v}^T = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} u_1 v_1 & u_1 v_2 & \cdots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \cdots & u_2 v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_m v_1 & u_m v_2 & \cdots & u_m v_n \end{bmatrix}$$