6.6 Under-determined Systems and RREF

As discussed in *Systems of Linear Equations*, under-determined systems of linear equations do not have a unique solution. If additional equations can **not** be found to make the system full rank, then one might try to find the set of points that satisfy the system of equations. Using *Elimination*, we can put the matrix into what is called **reduced row echelon form** or **RREF**, which will reveal the set of points where the system of equations is satisfied.

The word *echelon* just means resembling stair steps. A matrix is in row echelon form when:

- All nonzero values in each column are above all zeros.
- The leading coefficient of each row is strictly to the right of the leading coefficient in the row above it.

So when we perform Gauss-Jordan *Elimination* on an augmented matrix to make an upper-triangular matrix, we are putting the matrix into row echelon form.

An additional requirement for reduced row echelon form is:

- Every leading coefficient must be 1, and must be the only nonzero in its column. In RREF, the first m (number of rows) columns of the matrix should look as close as is possible to an identity matrix.

Thus a full rank matrix system in augmented form has the identity matrix in the first m columns when in RREF. The last column is the solution to the system of equations.

```
>> Ab = [A b]
Ab =
    8     1     6    27
    3     5     7    38
    4     9     2    55
>> rref(Ab)
ans =
    1     0     0     2
    0     1     0     5
    0     0     1     1
```

A b

$Ax = b$

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$x_1 = 2$, $x_2 = 5$, $x_3 = 1$

Now for an under-determined system:

```
>> A = randi(10, 3, 4) - randi(5, 3, 4)
A =
    6     0     4     0
    1     1     9     8
    5     9     5     5
>> rank(A)
ans =
    3
>> x = randi(5, 4, 1)
```

3x4

```
x =
     4
     2
     5
     1
>> b = A*x
b =
    44
    59
    68
>> C = rref([A b])
C =
    1.0000         0         0   -0.6091    3.3909
         0    1.0000         0    0.3864    2.3864
         0         0    1.0000    0.9136    5.9136
```

$Dx = e$

$x_1 \quad x_2 \quad D \quad x_3 \quad x_4 \quad | \quad e$

Notice that the first three columns, for x_1, x_2, and x_3, each have only a single one. But the fourth column, for x_4, has values in each row. MATLAB found the fourth column to be a linear combination of the first three columns by the weights indicated. So the values of the first three variables, which are pivot columns, are fixed by the equation. We call the variables associated with the pivot columns *basic variables*. The non-pivot column variable, x_4, is called a *free variable*, meaning that it can take any value.

We don't have an equation for x_4 since it is a free variable, so we can just say that $x_4 = x_4$. We can also replace it with an independent scalar variable, a.

The new system of equations is:

$$\begin{aligned} x_1 - 0.6091\,x_4 &= 3.3909 \\ x_2 + 0.3864\,x_4 &= 2.3864 \\ x_3 + 0.9136\,x_4 &= 5.9136 \\ x_4 &= x_4 \end{aligned}$$

$-$

Then solving for the unknown $\boldsymbol{x}$ vector,

$$\begin{aligned} x_1 &= 3.3909 + 0.6091\,a \\ x_2 &= 2.3864 - 0.3864\,a \\ x_3 &= 5.9136 - 0.9136\,a \\ x_4 &= x_4. \end{aligned}$$

$x = \begin{bmatrix} 4 \\ 2 \\ 5 \\ 1 \end{bmatrix}$

$a \doteq 1$

The set of solutions is a line. We can see this if we write the line equation in what is called parametric form.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3.3909 \\ 2.3864 \\ 5.9136 \\ 0 \end{bmatrix} + a \begin{bmatrix} 0.6091 \\ -0.3864 \\ -0.9136 \\ 1 \end{bmatrix}$$

Note: The solution to $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ is a combination of two solutions, a *general* solution, $\boldsymbol{u} = Null(\boldsymbol{A})$, and a *particular* solution, which is any vector that solves $\boldsymbol{A}\,\boldsymbol{v} = \boldsymbol{b}$. Let $\boldsymbol{x} = \boldsymbol{u} + \boldsymbol{v}$.

$$\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{A}\,(\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{0} + \boldsymbol{b} = \boldsymbol{b}$$

The general solution are any vectors that can be multiplied by any scalar. These vectors also form the basis vectors for the null space. The particular solution is the constant vector in the solution.

See *Null Space*.

Continuing the previous MATLAB commands, we can see that the $\boldsymbol{x}$ point that we used before is on a line, as is an infinite set of points. Some simple algebra shows that $a = 1$ yields $\boldsymbol{x}$ in the line equation.

```
>> D = C(:,5)
D =
    3.3909
    2.3864
    5.9136
>> E = -C(:,4)
E =
    0.6091
   -0.3864
   -0.9136
>> x4 = (x(1:3) - D)./E
x4 =
    1.0000
    1.0000
    1.0000
```

Here is an example with two free variables. Notice that in this case, the last row becomes all zeros, so it is not part of the solution. Here we begin with the augmented matrix.

```
>> Ab = [A b]
Ab =
     4     2     6    10     3    12
     8    -9    -1     7     5   -13
     2     3     5     7     4     7
    -2     5     3     1     3     3
>> C = rref(Ab)
C =
     1     0     1     2     0     3
     0     1     1     1     0     3
     0     0     0     0     1    -2
     0     0     0     0     0     0
```

A - 4x5

Dx = e

D

e

$x_1 + x_3 + 2x_4 = 3$

Not needed

$$\begin{aligned} x_1 &= 3 - x_3 - 2\,x_4 \\ x_2 &= 3 - x_3 - x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \\ x_5 &= -2 \end{aligned}$$

The solution is a hyperplane with two degrees of freedom in $x_3 = a$ and $x_4 = b$, which expressed in a parametric equation is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 0 \\ -2 \end{bmatrix} + a \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

We can see a plot of how the dependent variables relate to each other:

```
>> [a, b] = meshgrid(linspace(-20,20), linspace(-20,20));
>> X1 = 3 - a - 2*b;
>> X2 = 3 - a - b;
>> X5 = -2*ones(100);
>> surf(X5, X1, X2)
>> xlabel('X_5')
>> ylabel('X_1')
>> zlabel('X_2')
```

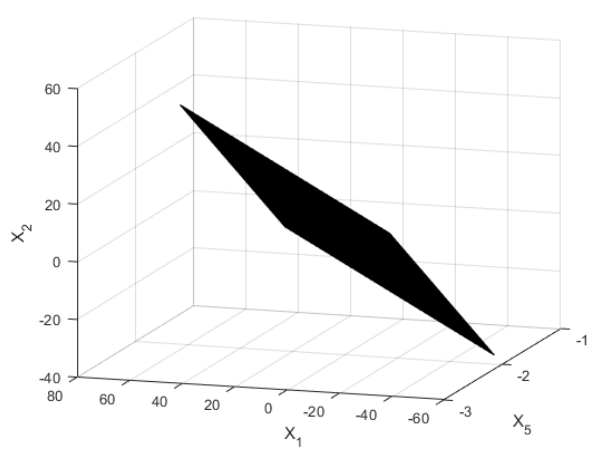

Fig. 6.9: The domain of solutions to the under determined system